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Perturbation of Zeros of Analytic Functions II

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I. INTRODUCTION

In part I of this paper [8] we examined the variation of the zeros of an analytic function f when f is varied by a small function, under the assumptions that f is bounded in the region considered and bounded away from zero at one point. No assumption was made regarding the multiplicities of the zeros of f.

In this part we study the behavior of zeros of specified multiplicity. To fix the ideas, we consider the Banach algebra R of all functions f, analytic and bounded in the unit circle $U: |z| < 1$, with the norm

$$\|f\| = \sup\{|f(z)| : z \in U\}.$$

Suppose that $f \in R$, $\|f\| \leqslant 1$ and that f has a zero of order n at 0. What can we say about the zeros of $g = f + \phi$, where $\phi \in R$ and $\|\phi\|$ is small?

We may represent f in the form

$$f(z) = z^n h(z)$$

where $h \in R$, $\|h\| \leqslant 1$, $|h(0)| = A > 0$. By Hurwitz' theorem we know that if $\|\phi\|$ is sufficiently small, then g has exactly n zeros near the origin. Our first concern is to make this statement precise and quantitative. We prove

THEOREM 2. *Under the above assumptions on f, if $\|\phi\| \leqslant \epsilon < \alpha_n(A)$, where*

$$\alpha_n(A) = \frac{n^n A^{n+1}}{(n+1)^{n+1}} \left(1 + \frac{nA^2}{n+1} + O(A^4)\right),$$

then the number $n(r, g)$ of zeros of g in $U_r: |z| < r$ is equal to n, for

$$\lambda_2 A < r < \lambda_1 A,$$

where λ_1 and λ_2 are the roots of the equation

$$\lambda^n(1 - \lambda) = \delta(1 - \lambda A^2), \qquad \epsilon = \delta A^{n+1}$$

in the interval $0 < \lambda < 1$.

As $\delta \to 0$ the numbers λ_1 and λ_2 satisfy

$$\lambda_1 = 1 - (1 - A^2)\,\delta + O(\delta^2)$$

and

$$\lambda_2 = \eta\left(1 + \frac{(1-A^2)}{n}\eta + O(\eta^2)\right), \qquad \eta^n = \delta.$$

While the proof follows classical lines, it is not usually observed that we can specify a small circle in which there are n zeros and a fairly large circle in which there are no other zeros.

For $n = 1$, we can obtain much more detailed information. If $z(g)$ is the (unique) smallest zero of g, then $z(g)$ is approximately $-g(0)/f'(0)$, and we have the error estimate

$$\left|z(g) + \frac{g(0)}{f'(0)}\right| \leqslant 12\epsilon^2/A^3.$$

Now the quantity $L(g) = -g(0)/f'(0) = -\phi(0)/f'(0)$ is a linear functional on R, and the estimate

$$z(g) - z(f) - L(g-f) = O(\|g-f\|^2)$$

shows that $L = z'(f)$ is the Fréchet derivative of z at f. The Fréchet differentiability of $z(g)$ means that $z(g)$ is an analytic function on a certain domain of R. Consequently, if g depends analytically on one or more parameters, then $z(g)$ is an analytic function of these parameters.

By slight modifications of the argument, we obtain

COROLLARY 1e. *The function $z(f)$, the smallest zero of f, is defined and analytic in the domain $\mathfrak{D}$ of R defined by*

$$\mathfrak{D}\colon \|f\| < 1, \qquad |f'(0)|^2 > 4|f(0)|(1-|f(0)|^2)^2,$$

and its Fréchet derivative is

$$z'(f)(\phi) = -\frac{\phi(z(f))}{f'(z(f))}.$$

The problem of obtaining the higher variations, i.e., the higher Fréchet derivatives, of $z(f)$ on $\mathfrak{D}$, turns out to be equivalent to that of finding the classical Lagrange expansion (Whittaker–Watson [9], p. 132) of the smallest zero of

$$z - \lambda\psi(z) = 0$$

in powers of λ. We can also attain error estimates for the power series expansion of $z(f + \lambda\phi)$.

The approximation formula

$$z(g) \sim z - \frac{g(z)}{g'(z)} = H(z) = H(z, g)$$

if z is close to $z(g)$, which is implied by the formula in the above corollary, is the first step in Newton's method for functions of the class considered.

In the appendix we prove

THEOREM 6. *Let $E(A)$ be the set of $f \in R$ such that $f(0)=0$, and $|f'(0)|=A$. Let $H(z)=H(z,f)$ and $0<k\leqslant 1$. Then $|H(z)|\leqslant k|z|$ for $|z|\leqslant r(k)$, where*

$$r(k)=A(\sigma-1)/(\sigma+2k+1)$$

and

$$\sigma^2=((2k+1)^2-A^2)/(1-A^2).$$

This is the best possible, and is attained only by $f(z)=\mathfrak{F}(cz)$, $|c|=1$, where

$$\mathfrak{F}(z)=z\left(\frac{A-z}{1-Az}\right).$$

The iterates of H converge to 0 in the circle $|z|<r(1)$.

If $H(x)$ is real and nonnegative on the interval $0\leqslant x\leqslant A$, then the iterates of H converge on the interval $0\leqslant x<A/(2-A)^2$, and this is also the best possible. While $r(1)$ is the radius of the largest circle in which H is a contraction for all $f\in E(A)$, we do not know whether this is the largest circle in which the iterates of H converge to zero. For references to the literature on Newton's method, see Ostrowski [7].

In part I, we derived a criterion of the form

$$|f(0)|<B(|z_1|,|f(z_1)|) \qquad \text{if } \|f\|\leqslant 1, f\in R,$$

for the existence of a small zero of f. We also have the criteria

$$2|f(0)|\log(1/|f(0)|)<|f'(0)|$$

for the existence of a small zero, and

$$4|f(0)|(1-|f(0)|^2)^2<|f'(0)|^2$$

for the existence and uniqueness of a small zero, expressed in terms of $|f(0)|$ and $|f'(0)|$. In the appendix, we give a criterion for the existence of a small zero in terms of the values of $|f(0)$ and $|f(z_1)|$, where z_1 depends on $f(0)$.

The simplest example is

COROLLARY 5a. *If $f\in R$, $\|f\|\leqslant 1$, $|z_1|\leqslant|f(0)|$, and if $f(z)\neq 0$ in the circle $|z|<|z_1|/t_0$, where*

$$t_0=|\zeta-\alpha|/(\zeta+\alpha),$$

$$\zeta=\log|1/f(z_1)|, \qquad \alpha=\log(1/|f(0)|),$$

then

$$-2\eta \leqslant \alpha - \zeta \leqslant 2$$

where $\eta \exp(1+\eta) = 1$, $\eta > 0$.

Thus if $\log|f(z_1)/f(0)|$ is greater than 2 or less than -2η, f has a small zero. In particular, we may take $z_1 = f(0)$, and we obtain the criterion that if $f(f(0))/f(0)$ is too large or too small, then f has a small zero.

If $n > 1$, then the n-tuple zero of f splits, in general, into n small zeros of g, and these have an algebraic singularity at $g = f$. However, we can represent g in the form

$$g(z) = P(z, g)\, G(z, g),$$

where

$$P(z, g) = z^n + Q(z, g),$$

Q is a polynomial of degree $<n$ in z, and G is analytic in z and $\neq 0$ in a neighborhood of zero. Furthermore, P and G are analytic functions of g for $\|g - f\|$ sufficiently small.

This is the essential content of the Weierstrass preparation theorem (see Bochner–Martin [2], p. 183). We give a proof which yields explicit estimates for the various quantities and domains involved. (A similar proof was given in the thesis of Brown [3].)

If $g = f + \lambda\phi$, $Q = Q(z, \lambda)$, $G = G(z, \lambda)$,

then

$$Q_\lambda(z, 0) = s_{n-1}(z, \phi/h),$$

and

$$G_\lambda(z, 0) = h(z)\, R_n(z, \phi/h),$$

where

$$s_n\left(z, \sum_0^\infty a_k z^k\right) = \sum_0^n a_k z^k,$$

and

$$R_n(z, F) = (F(z) - s_{n-1}(z, F))/z^n.$$

We obtain explicit estimates for

$$|Q(z, 1) - Q_\lambda(z, 0)|$$

and

$$|G(z, 1) - 1 - G_\lambda(z, 0)|$$

in terms of $\|\phi\|$.

The Weierstrass theorem is a special case of a general theorem on Banach algebras. We prove

THEOREM 4. *If M is a closed module in a Banach algebra B, and $R(M)$ is the set of $x \in B$ such that every $y \in B$ has a unique representation in the form*

$$y = qx + r, \qquad q \in B, \qquad r \in M,$$

then $R(M)$ is open. The solutions $q = S_x(y)$ and $r = T_x(y)$ are bounded linear transformations on B into B and M, respectively, and S_x and T_x are analytic functions of x in $R(M)$. Specifically, the sphere $\|\xi - x\| < 1/\|S_x\|$ is contained in $R(M)$.

In this sphere we can obtain the Fréchet derivatives of S_x and T_x, and estimates of the form

$$\|S_\xi(y) - S_x(y) + S_x(S_x(y)(\xi - x))\| \leqslant C_1 \|\xi - x\|^2 \|y\|$$

and

$$\|T_\xi(y) - T_x(y) + T_x(S_x(y)(\xi - x))\| \leqslant C_2 \|\xi - x\|^2 \|y\|.$$

The Weierstrass theorem is the simple special case $B = R$, $M =$ the set of polynomials of degree $<n$, and $x = z^n$. In this case, if $y \in R$, then

$$S_x(y)(z) = R_n(z, y)$$

and

$$T_x(y)(z) = s_{n-1}(z, y).$$

In part III of this paper, we study the simultaneous variation of all the zeros of $f \in R$ in a compact subset of U, under perturbation of f.

II. Perturbation of Simple Zeros

Let $f \in R$, $\|f\| \leqslant 1$, $f(0) = 0$, $f'(0) = a$. If $g \in R$, and $\|g - f\|$ is sufficiently small, then g has a unique zero $z(g)$ near 0. We wish to study $z(g)$ in some detail.

Let $f(z) = zh(z)$, where $h \in R$, $\|h\| \leqslant 1$, and $h(0) = a$, and let

$$u(z) = (h(z) - a)/(1 - \bar{a}h(z)).$$

Since $\|u\| \leqslant 1$, $u(0) = 0$, then, by Schwarz' lemma, $|u(z)| \leqslant |z|$. From

$$h = (u + a)/(1 + \bar{a}u),$$

we obtain for $|z| \leqslant r < A = |a|$,

$$|h(z)| \geqslant (A - r)/(1 - Ar)$$

and

$$|f(z)| \geqslant r(A-r)/(1-Ar) = k(r).$$

Hence, by Rouché's theorem, if $0 < r < A$ and

$$\|g-f\| \leqslant \epsilon < k(r),$$

then g has a unique zero $z(g)$ in the circle U_r: $|z| < r$.

The function $k(r)$ attains its maximum $\alpha(A)$ in the interval $(0, A)$ at

$$\begin{aligned} r = r(A) &= (1-(1-A^2)^{1/2})/A \\ &= \frac{A}{2} + \frac{A^3}{8} + \cdots + \frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{2^n n!} A^{2n-1} + \cdots \end{aligned} \tag{1}$$

and

$$\begin{aligned} \alpha(A) = k(r(A)) &= -1 + 2r(A)/A \\ &= \frac{A^2}{4} + \frac{A^4}{8} + \cdots. \end{aligned}$$

If $0 < \epsilon < \alpha(A)$, then the equation

$$k(r) = \epsilon$$

has two roots in (U, A):

$$r_2(\epsilon) = \epsilon^{1/2} r(\delta) < r_1(\epsilon) = \epsilon^{1/2}/r(\delta) < A,$$

where

$$\delta = 2\epsilon^{1/2}/A(1+\epsilon).$$

Consequently, if $\|g-f\| \leqslant \epsilon < \alpha(A)$, then g has a unique zero $z(g)$ in the circle $|z| < r_1(\epsilon)$, and

$$|z(g)| \leqslant r_2(\epsilon).$$

The estimates

$$\frac{A}{2} \leqslant \frac{A}{2} + \frac{A^3}{8} \leqslant r(A) \leqslant \frac{A}{2} + \frac{A^3}{2} \leqslant A,$$

and

$$A^2/4 \leqslant \alpha(A) \leqslant A^2,$$

are often sufficiently accurate. Thus, we have

$$|z(g)| \leqslant 2\epsilon/A(1+\epsilon) \leqslant 2\epsilon/A$$

and if $\epsilon \leqslant A^2/4$, then g has no zeros in the annulus

$$2\epsilon/A(1+\epsilon) < |z| < A(1+\epsilon)/(1+\delta^2).$$

Hence, we obtain

THEOREM 1. *If* $f \in R$, $\|f\| \leqslant 1$, $f(0) = 0$, $|f'(0)| = A \neq 0$, *and if* $r(A)$, $\alpha(A)$, $r_1(\epsilon)$, *and* $r_2(\epsilon)$ *are defined as above, then for* $g \in R$, *and*

$$\|g - f\| \leqslant \epsilon < \alpha(A),$$

g has a unique zero $z(g)$ *in the circle* $|z| < r_1(\epsilon)$ *and*

$$|z(g)| \leqslant r_2(\epsilon).$$

As $\epsilon \to 0$, we have

$$r_2(\epsilon) \sim \epsilon/A, \qquad A - r_1(\epsilon) \sim \epsilon(1 - A^2)/A,$$

and as $\epsilon \to \alpha(A)$, we have

$$\lim r_1(\epsilon) = \lim r_2(\epsilon) = r(A).$$

Let $F(z) = g(z) - az - g(0)$, where $a = f'(0) = b(0)$. If we apply the Schwarz lemma to $g(z) - f(z) - g(0)$, we obtain

$$|g(z) - f(z) - g(0)| \leqslant 2\epsilon |z|.$$

Similarly, we have

$$|h(z) - a| \leqslant 2|z|,$$

so that

$$F(z) = g(z) - f(z) - g(0) + z(h(z) - a)$$

satisfies

$$|F(z)| \leqslant 2|z|^2 + 2\epsilon |z|.$$

If $\epsilon < \alpha(A)$ and $z = z(g)$, we have

$$|az + g(0)| \leqslant 2|z|^2 + 2\epsilon|z| \leqslant 12\epsilon^2/A^2.$$

This yields

COROLLARY 1a. *Under the hypotheses of Theorem* 1,

$$|z(g) + g(0)/f'(0)| \leqslant 12\epsilon^2/A^2.$$

Therefore $z(g)$ *is Fréchet differentiable at* f, *and*

$$z'(f)(g) = -g(0)/f'(0).$$

If we apply this result to $g(z) = f(z) - w$, where w is a constant, we obtain

COROLLARY 1b. *For* $0 < \epsilon \leqslant \alpha(A)$, *the image of the circle* U_r, $r = r_2(\epsilon)$, *contains the circle* U_ϵ. *In particular, the circle* $U_{\alpha(A)}$ *is contained in the range of* f. *The inverse function* f^{-1} *is defined in* $U_{\alpha(A)}$,

$$|f^{-1}(w)| \leqslant r_2(|w|) \leqslant \frac{|w|}{A}\left(1 + \frac{4|w|}{A^2}\right),$$

and

$$\left|f^{-1}(w) - \frac{w}{f'(0)}\right| \leqslant 12|w|^2/A^3.$$

COROLLARY 1c. *If $f \in R, f(0) = 0, f'(0) = 1,$ and $\|f\| \leqslant M$, then the range of f contains the circle U_r, $r = 1/4M$.*

This is obtained by applying the previous corollary to $F = f/M$, and using the estimate $\alpha(A) \geqslant A^2/4$.

Corollary 1c is the main step in the classical proof of Bloch's theorem (see Landau [*6*]). If we apply the same reasoning, using Theorem 1 instead of the corollary, we obtain

COROLLARY 1d. *If $f \in R, f'(0) = 1$, then there is a constant c such that $f + \phi - c$ has a zero in U for all $\phi \in R$ such that $\|\phi\| \leqslant 1/16$.*

If $f \in R$, $\|f\| \leqslant 1, f(0) = a_0, f'(0) = a_1$, then $f = \phi(F)$, where

$$\phi(w) = \frac{a_0 + w}{1 + \bar{a}_0 w},$$

$$F \in R, \qquad \|F\| \leqslant 1, \qquad F(0) = 0$$

and

$$F'(0) = \frac{a_1}{1 - (a_0)^2}.$$

The above results, applied to F and to $g = a_0 + F$, imply that if $4|a_0|(1 - |a_0|^2)^2 < |a_1|^2$, then f has a unique zero $z(f)$ in the circle U_r, $r = r_1(|a_0|, B), B = |a_1|/(1 - |a_0|^2)$, and that

$$|z(f)| \leqslant \frac{|a_0|}{B_1}\left(1 + \frac{4|a_0|}{B_1}\right). \tag{1}$$

A weaker sufficient condition on $f \in R$, $\|f\| \leqslant 1$, for the existence of a small zero with, however, a cruder estimate of $|z(f)|$ is:

If $f(0) = \epsilon > 0$, $|f'(0)| = A$, and $2\epsilon \log(1/\epsilon) < A$, then f has a zero in U_r, where

$$r = 2\epsilon \log(1/\epsilon)/A = 2\gamma(f).$$

This is obtained by applying Schwarz' lemma to

$$f_1(z) = \frac{\log \epsilon - \log f(z)}{\log \epsilon + \log f(z)}$$

in the circle U_r, where $r = |z(f)|$.

The example $f(z) = ((z + \delta)/(1 + \delta))^2$, $0 < \delta < 1$, shows that $\gamma(f)$ can be arbitrarily small and $|f'(0)|^2/|f(0)|$ can be arbitrarily close to 4 for a function with a small multiple zero. Therefore, no condition of the above type, on $\gamma(f)$, can insure the uniqueness of $z(f)$, and the constant 4 in the above results is the best possible.

A complex valued function $T(f)$ on R is said to be analytic at f_0 if it is defined at f_0 and if there is a linear functional L on R such that

$$|T(f_0+\phi)-T(f_0)-L(\phi)| = o\|\phi\|)$$

for $\phi \in R$, $\|\phi\| \to 0$. We call $L = T'(f_0)$ the Fréchet derivative of T at f_0.

The above results imply

COROLLARY 1e. *The function $z(f)$, the smallest zero of f, is defined and analytic in the domain of R defined by*

$$\mathfrak{D}: \|f\| < 1,\ |f'(0)|^2 > 4|f(0)|(1-|f(0)|^2)^2,$$

and its Fréchet derivative is

$$z'(f)(\phi) = -\frac{\phi(z(f))}{f'(z(f))}.$$

To obtain the higher variations of $z(f)$, we must study the smallest zero of $g = f + \lambda\phi$ as a function of λ. If

$$f \in R, \qquad \|f\| \leqslant 1, \qquad f'(0) = a \neq 0,$$

and

$$Z(\lambda) = z(f + \lambda\phi),$$

where $\phi \in R$, then the nth Fréchet derivative $z^{(n)}(f)(\phi)$ is given by

$$z^{(n)}(f)(\phi) = Z^{(n)}(0)$$

and we have the power series expansion

$$z(f+\phi) = \sum_0^\infty z^{(n)}(f)(\phi)/n!.$$

But $Z(\lambda)$ is the smallest zero of

$$z - \lambda\psi(z) = 0,$$

where $\psi = -\phi/h$. We arrive at the classical Lagrange expansion (see Whittaker–Watson [9]). The Cauchy formula yields

$$Z(\lambda) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{z(1-\lambda\psi'(z))\,dz}{z-\lambda\psi(z)},$$

where $\mathscr{C}$ is the circle $|z| = r$, and $r_2(\epsilon) < r < r_1(\epsilon)$, $\epsilon = \|\phi\|$, $|\lambda| \leqslant 1$. On $\mathscr{C}$ we have

$$|\psi(z)| \leqslant \epsilon/k(r),$$

so that the most favorable choice of r is $r = r(A)$, and then $k(r) = \alpha(A)$. Since

$$z(1 - \lambda\psi'(z)) = z - \lambda\psi(z) + \lambda(\psi(z) - z\psi'(z)),$$

we see that

$$Z(\lambda) = \frac{1}{2\pi i}\int_{\mathscr{C}} zu'(z)\,dz = -\frac{1}{2\pi i}\int_{\mathscr{C}} u(z)\,dz,$$

where $u(z) = \log(1 - \lambda\psi(z)/z)$, taking the determination such that $|\arg u(z)| < \pi/2$ for $z \in \mathscr{C}$. We thus obtain

$$Z(\lambda) = \sum_0^\infty c_n \lambda^n,$$

where

$$c_n = \frac{1}{2\pi i n}\int_{\mathscr{C}} \left(\frac{\psi(z)}{z}\right)^n dz = \frac{1}{n!}(D^{n-1}\psi^n)(0)$$

and $D = d/dz$. This yields

$$z^{(n)}(\phi) = (-1)^n (D^{n-1}(\phi/h)^n)(0). \tag{2}$$

The error

$$\epsilon_n = \left|z(f+\phi) - \sum_0^n z^{(k)}(f)(\phi)/k!\right| \tag{3}$$

can be estimated by

$$\epsilon_n \leqslant \frac{1}{n+1}\left(\frac{\epsilon}{\alpha}\right)^{n+1}\frac{1}{1-\epsilon/\alpha}, \tag{4}$$

if $\alpha = \alpha(A)$, $0 < \epsilon < \alpha(A)$.

We remark that by using the facts that $Z(\lambda)$ is analytic and $|Z(\lambda)| \leqslant r(A)$ for $|\lambda| < \alpha/\epsilon$, and by applying Landau's theorem (see Landau [*6*], p. 26), we can prove that

$$\epsilon_n \leqslant (K_n + 1)\, r(A)(\epsilon/\alpha)^{n+1}, \tag{5}$$

where

$$K_n = \sum_0^n \binom{-1/2}{k}^2,$$

$$K_n \sim \log n/\pi \qquad \text{as } n \to \infty.$$

Thus, we obtain

COROLLARY 1f. *If $f \in \mathfrak{D}$, $f(0) = 0$, $\phi \in R$, $\|\phi\| \leqslant \epsilon < \alpha = \alpha(A)$, then the* nth *Fréchet derivative of $z(f)$ is given by* (2), *and the error ϵ_n in the approximation* (3) *can be estimated by* (4) *and* (5).

III. Perturbation of Multiple Zeros

Let $f \in R$, $\|f\| \leqslant 1$, $f(z) = z^n h(z)$, where $h \in R$, $\|h\| \leqslant 1$, $h(0) = A > 0$, and $n > 1$. If $g \in R$, and $\|g - f\|$ is small, then g has n zeros near 0. We wish to study these zeros in some detail.

As before, we have

$$|f(z)| \geqslant r^n(a - r)/(1 - Ar) = k_n(r)$$

for $|z| \leqslant r < A$. Hence, if

$$\|g - f\| \leqslant \epsilon < k_n(r),$$

then g has exactly n zeros in the circle U_r.

Now, $k_n(r)$ attains its maximum in $(0, A)$ at

$$\begin{aligned} r = r_n(A) &= [b - (b^2 - 4)^{1/2}]/2 \\ &= \frac{\mu A^2 + 1 - [(1 - A^2)(1 - \mu^2 A^2)]^{1/2}}{A(1 + \mu)} \\ &= \frac{n}{n+1} A + \frac{nA^3}{(n+1)^3} + \ldots, \end{aligned}$$

where $b = [n(1 + A^2) + 1 - A^2]/nA$, $\mu = (n - 1)/(n + 1)$. This radius $r_n(A)$ is, thus, expressed as a power series in A with nonnegative coefficients, which easily yields the estimates:

$$\frac{nA}{n+1} < \frac{nA}{n+1} + \frac{nA^3}{(n+1)^3} < r_n(A) < \frac{nA + A^3}{n+1} < A.$$

The maximum

$$\begin{aligned} \alpha_n(A) &= k_n(r_n(A)) \\ &= r_n(A)^n (nr_n(A) - (n-1) A) \\ &= \frac{n^n A^{n+1}}{(n+1)^{n+1}} \left(1 + \frac{nA^2}{n+1} + O(A^4)\right) \end{aligned}$$

satisfies

$$\frac{r_n(A)^{n+1}}{n} < \alpha_n(A) < r_n(A)^{n+1}.$$

If $0 < \epsilon < \alpha_n(A)$, then the equation

$$k_n(r) = \epsilon$$

has two roots $0 < \rho_2 < \rho_1$ in the interval $(0, A)$. If we set $\epsilon = \delta A^{n+1}$, $r = \lambda A$, we find that λ satisfies

$$\lambda^n (1 - \lambda) = \delta(1 - \lambda A^2), \qquad 0 < \lambda < 1. \tag{6}$$

For small δ, we see that the two solutions $0 < \lambda_2 < \lambda_1 < 1$ satisfy

$$\lambda_2 = \eta \left(1 + \frac{(1 - A^2)}{n} \eta + O(\eta^2)\right), \qquad \eta^n = \delta,$$

and

$$\lambda_1 = 1 - (1 - A^2)\,\delta + O(\delta^2).$$

A little computation shows that if $\eta \leqslant 1/6$, then

$$\lambda_2 \leqslant \eta \left(1 + \left(\frac{1 - A^2}{n}\right) \eta(1 + 6\eta)\right).$$

We obtain the result:

THEOREM 2. *If $f \in R$, $\|f\| \leqslant 1$, $f(z) = z^n h(z)$, where $|h(0)| = A > 0$, and if $g \in R$, $\|g - f\| \leqslant \epsilon < \alpha_n(A)$, then the number $n(r, g)$ of zeros of g in U_r satisfies $n(r, g) = n$ for*

$$\lambda_2 A < r < \lambda_1 A,$$

where λ_1 and λ_2 are the roots of (6) *in* (0, 1).

We remark that while the above is the best possible condition on $\|g - f\|$, the behavior of g outside U_A is not used in the above proof. If $M(r, F) = \sup\,(|F(z)| : |z| \leqslant r)$ (so that $\|F\| = M(1, F)$), then it suffices for the above conclusion that

$$M(A, g - f) \leqslant \epsilon < \alpha_n(A).$$

We can apply these remarks to obtain a generalization of Corollary 1e:

COROLLARY 2a. *If $f \in R$, $\|f\| \leqslant 1$,*

$$f(z) = \sum_0^\infty a_k z^k, \qquad s_n(z) = \sum_0^n a_k z^k,$$

and

$$M(A, s_{n-1}) = \epsilon < K_n{}' \alpha_n(A),$$

where

$$A = |a_n|/K_n{}', \qquad K_n{}' = K_{n-1} + 1$$

and K_n is the constant in (5), *then $n(r, f) = n$ for*

$$\lambda_2(\epsilon/K_n{}')\,A < r < \lambda_2(\epsilon/K_n{}')\,A.$$

For we have

$$f(z) = s_{n-1}(z) + z^n h(z),$$

where $h \in R$, $h(0) = a_n$, and $\|h\| = \|f - s_{n-1}\| \leqslant K_n'$. Hence theorem 2 applies to $g = f/K_n'$.

For $n = 2$, we have $\alpha_n(A) > 4A^3/9$, and $K_2' = 9/4$, which leads us to the condition

$$\epsilon = |a_0| + 4|a_1|\,|a_2|/9 < (4|a_2|/9)^3 = A^3$$

for f to have two small zeros. If $\delta = 4\epsilon/9A^3$, then the two zeros are in $|z| < \lambda_2 A$, and there are no other zeros in $|z| < \lambda_1 A$, where $0 < \lambda_2 < \lambda_1$ are the roots in $(0, 1)$ of the equation

$$\lambda^2(1 - \lambda) = \delta(1 - A^2\lambda).$$

Since the nth-order zero of $f = z^n h$ splits, in general, into n zeros of g when $\|g - f\|$ is small, $z(g)$ has a branch point at $g = f$, and this algebraic singularity is rather complicated. If $z_1, \ldots, z_n$ are the small zeros of g, then

$$P(z, g) = \prod_1^n (z - z_k) = z^n + Q(z), \deg Q < n,$$

is an analytic function of g. We have the representation

$$g = PG,$$

where $G \in R$ and $G(z) \neq 0$ in a neighborhood of 0 containing $z_1, \ldots, z_n$. This is essentially the Weierstrass preparation theorem.

If $f = z^n h$, $h(0) \neq 0$, then we can easily reduce the study of the representation of $g = f + \phi$, $\|\phi\|$ small, in the form PG, to the special case $h \equiv 1$. We shall, therefore, first analyze the problem:

For $\phi \in R$, $\|\phi\| \leqslant \epsilon$, find a polynomial $P = z^n + Q$, $\deg Q < n$, and a function G, such that G, $1/G \in R$, and

$$g = z^n + \phi = PG.$$

We wish to obtain control over the dependence of P and G on ϕ.

Of course, if $\|\phi\| \leqslant \epsilon < 1$, then we see, by Roché's theorem, that $n(r, g) = n$ for $\epsilon^{1/n} < r \leqslant 1$. We have the formula

$$\begin{aligned} \log(P(z)/z^n) &= \log(1 + Q(z)/z^n) \\ &= \Lambda(z, g) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{g'(\zeta)}{g(\zeta)} \log\left(1 - \frac{\zeta}{z}\right) d\zeta, \end{aligned}$$

where $\mathscr{C}$ is the circle $|\zeta| = r$, $\epsilon^{1/n} < r < 1$, and $|z| > r$. This shows that Λ, and $P = z^n \exp \Lambda$, are analytic in the sphere $\|g - z^n\| < 1$ in R.

We can put the formula for Λ in another form which may be useful for some purposes. Let $g(z, \lambda) = z^n + \lambda\phi(z)$, where $\|\phi\| \leqslant \epsilon < 1$, and $|\lambda| \leqslant 1$, and let

$$\psi(z, \lambda) = \log(g(z, \lambda)/z^n).$$

Then ψ is analytic in λ and z for $|\lambda| \leqslant 1$, $\epsilon^{1/n} < |z| < 1$, and

$$\frac{g_z(z,\lambda)}{g(z,\lambda)} = \frac{n}{z} + \psi_z(z,\lambda).$$

Therefore

$$\begin{aligned} \Lambda(z,g) &= \frac{1}{2\pi i}\int_{\mathscr{C}} \psi_\zeta(\zeta,\lambda)\log\left(1-\frac{\zeta}{z}\right)d\zeta \\ &= \frac{1}{2\pi i}\int_{\mathscr{C}} \psi(\zeta,\lambda)\frac{d\zeta}{z-\zeta} \\ &= \sum_1^\infty c_k(z)\lambda^k, \end{aligned} \tag{1}$$

where

$$\begin{aligned} c_k(z) &= \frac{(-1)^{k-1}}{2\pi ik}\int_{\mathscr{C}} \frac{\phi(\zeta)^k\,d\zeta}{\zeta^{nk}(z-\zeta)} \\ &= \frac{(-1)^{k-1}}{kz^{nk}}\, s_{nk-1}(z,\phi^k), \end{aligned}$$

and

$$s_m\left(z, \sum_0^\infty a_k z^k\right) = \sum_0^m a_k z^k.$$

Of course

$$k!\,c_k(z) = \left(\frac{\partial}{\partial\lambda}\right)^k \Lambda(z, z^n+\phi)|_{\lambda=0} = \Lambda^{(k)}(z,z^n)(\phi)$$

is the kth Fréchet derivative (kth variation) of Λ at the point z^n in R. In particular, we have the first variation of P:

$$\begin{aligned} P^{(1)}(z,z^n)(\phi) = Q^{(1)}(z,z^n)(\phi) &= z^n\Lambda^{(1)}(z,z^n)(\phi) \\ &= s_{n-1}(z,\phi), \end{aligned}$$

and the estimate

$$|P(z,g) - z^n - s_{n-1}(z,\phi)| \leqslant B\epsilon^2, \qquad |z| \leqslant 1, \tag{2}$$

for a certain constant B.

While we can obtain an estimate for B directly from the formula for Λ, another approach is also quite instructive. Since we have

$$P(z) = \prod_1^n (z - z_k), \qquad |z_k| \leqslant \epsilon^{1/n},$$

then for $|z| = 1$,

$$(1-\epsilon^{2/n})^n \leqslant |P(z)| \leqslant (1+\epsilon^{1/n})^n \leqslant 2^n,$$

and therefore

$$|G(z)^{-1}| = |P(z)/g(z)| \leqslant 2^n/(1-\epsilon).$$

Since $G(z)^{-1} \in R$, this inequality also holds for $|z| < 1$, i.e., $\|G^{-1}\| \leqslant 2^n/(1-\epsilon)$. Let $H(z) = 1/G(z) = 1 + H_1(z)$. Then the equation

$$P = z^n + Q = (z^n + \phi)\,H$$

implies

$$Q = z^n H_1 + H\phi,$$

or

$$Q = s_{n-1}(z, H\phi).$$

Hence, by Landau's theorem, we obtain

$$\|Q\| \leqslant K_{n-1}\|H\|\,\|\phi\| \leqslant K_{n-1}\,2^n\,\epsilon/(1-\epsilon).$$

Moreover, if we define the operator Γ_2 by

$$\Gamma_2\left(\sum_0^\infty a_k z^k\right) = \sum_2^\infty a_k z^{k-n},$$

then we have

$$H_1 + \Gamma_2(H\phi) = 0. \tag{3}$$

But for $\|f\| \leqslant 1$, we have

$$|(\Gamma_2 f)(z)| \leqslant |(f(z) - s_{n-1}(z))/z^n| \leqslant K_n',$$

i.e., $\|\Gamma_2\| \leqslant K_2'$. Hence we infer that

$$\|H_1\| \leqslant K_n'\|H\|\,\|\phi\| \leqslant K_n'\,2^n\,\epsilon/(1-\epsilon),$$

and therefore

$$\begin{aligned} |Q - s_{n-1}(z,\phi)| &= |s_{n-1}(z, H_1\,\phi)| \\ &\leqslant K_{n-1}\,K_2'\,2^n\,\epsilon^2/(1-\epsilon). \end{aligned}$$

This yields (2), with

$$B = K_{n-1}\,K_n'\,2^n/(1-\epsilon). \tag{4}$$

Higher order approximations can be obtained by solving

$$H_1 = -\Gamma,\phi - \Gamma_2(\phi H_1)$$

by iteration, and substituting the results in (3).

Since P is a perturbation of $P_0 = z^n + s_{n-1}(z, \phi)$ of the order of $O(\epsilon^2)$, we see, by the results of this and the preceding section, that if z_0 is a zero or order m of P_0, then P has m zeros in a circle of radius $O(\epsilon^{2/m})$ about z_0.

We may summarize these results as follows:

THEOREM 3. *If* $\phi \in R$, $\|\phi\| \leqslant \epsilon < 1$, *and* $g = z^n + \phi$, *then* $n(r,g) = n$ *for* $\epsilon^{1/n} < r \leqslant 1$, *and there is a polynomial* $Q = Q(z,g)$, $\deg Q < n$, *and a function* $G = G(z,g)$ *such that*

$$g = (z^n + Q)\,G \quad \text{in } U,$$

G, $1/G \in R$.

The function $P = z^n + Q = z^n \exp \Lambda$ is analytic in g in the sphere $\|\phi\| < 1$ in R, and so is G. The function $\Lambda = \Lambda(z,g)$ is given by the formula (1) (with $\lambda = 1$). The first variation of P is $s_{n-1}(z,\phi)$, and the error is estimated by (2) and (4).

The function G satisfies

$$(1-\epsilon)/2^n \leqslant |G(z)| \leqslant (1+\epsilon)/(1-\epsilon^{1/n})^n,$$

$$\|1 - 1/G\| \leqslant K_n' \, 2^n \epsilon/(1-\epsilon),$$

and

$$\|G - 1\| \leqslant K_n' \, 2^n \epsilon(1+\epsilon)/(1-\epsilon)(1-\epsilon^{1/n})^n.$$

More generally, if $f \in R$, $\|f\| \leqslant 1$, and $f(z) = z^n h(z)$, where $h \in R$, $|h(0)| = A > 0$, and $\phi \in R$, $\|\phi\| \leqslant \epsilon$, then, for $g = f + \phi$, we have

$$g(rz)/r^n = h(rz)(z^n + \phi_1(z)),$$

where

$$\phi_1(z) = \phi(rz)/r^n h(rz).$$

If $0 < r < A$, then $\phi_1 \in R$, and

$$\|\phi_1\| \leqslant \epsilon(1 - Ar)/r^n(A - r) = \epsilon/k_n(r),$$

so that the most favorable choice of r is $r_n(A)$, and then $\|\phi_1\| \leqslant \epsilon/\alpha_n(A)$.

Therefore, if $\epsilon < \alpha_n(A)$, we have the factorization

$$g(rz)/r^n = (z^n + Q_1)\,G_1,$$

where Q_1 is a polynomial of degree $< n$, and G_1, $1/G_1 \in R$. Hence we have

$$g(z) = (z^n + r^n Q_1(z/r))\,G_1(z/r) = (z^n + Q(z))\,G(z),$$

where Q is a polynomial of degree $< n$, and G and $1/G$ are bounded and analytic in U_r, $r = r_n(A)$. Furthermore, Q and G are analytic functions of g in the sphere $\|g - f\| < \alpha_n(A)$ of R. Thus, if g depends analytically on some parameters, then Q and G will be analytic functions of these parameters.

In particular, we obtain the classical case of the Weierstrass preparation theorem:

COROLLARY 3a. *If* F *is analytic on* $U \times U$, $\|F\| \leqslant 1$, and $F(z,0) = z^n h(z)$, *where* $h \in R$, $|h(0)| = A > 0$, *then there exist* $Q = Q(z,t)$ *and* $G = G(z,t)$ *such that* Q *is* *a polynomial of degree* $<n$ *in* z, *and* Q, G, *and* $1/G$ *are analytic in* $|t| < \alpha_n(A)/2$, *and* $|z| \leqslant r_n(A)$,

$$F(z,t) = (z^n + Q(z,t))\,G(z,t).$$

Furthermore, if $F_t(z,0) = f_1(z)$, *then we have,*

$$Q_t(z,0) = s_{n-1}(z, f_1/h)$$

and

$$G_t(z,0) = h(z)(\Gamma_2(f_1/h))(z).$$

Of course, we can easily give bounds on Q, G, and $1/G$ in the bicylinder $U_r \times U_s$, $r = r_n(A)$, $s = \alpha_n(A)/2$.

One of the main values of the above results concerning the analyticity of Q and G, and giving bounds on them, is that they are uniform in the sphere $\|\phi\| \leqslant \epsilon$ and do not depend on any more detailed information regarding ϕ. If $\mathfrak{I}_k$ is the ideal in R generated by z^k, i.e., the set of $\phi \in R$ which have a zero of multiplicity $\geqslant k$ at 0, then for $\phi \in \mathfrak{I}_k - \mathfrak{I}_{k+1}$, we can obtain another proof of theorem 3 and another representation of P and G, by using the approach of Corollary 1f.

Let us assume, for the sake of simplicity, that $h \equiv 1$. (We have seen how to reduce the general case to this special case.) If $\phi \in \mathfrak{I}_k - \mathfrak{I}_{k-1}$, then ϕ can be represented in the form $\phi = -z^k \psi^{n-k}$, and this representation is unique, if we restrict $\arg\psi(0)$ in an obvious way. Since the case $k \geqslant n$ is trivial, we may assume that $k < n$.

Then we have, setting $\lambda = \mu^{n-k}$,

$$\begin{aligned} g = z^n + \lambda\phi &= z^k(z^{n-k} - \mu^{n-k}\psi^{n-k}) \\ &= z^k \prod_1^{n-k} (z - \omega^j \mu\psi(z)), \end{aligned}$$

where

$$\omega = \exp(2\pi i/(n-k)).$$

But the equation

$$z - t\psi(z) = 0$$

has a unique solution $z = \zeta(t)$ such that $\zeta(0) = 0$ and ζ is analytic for $|t| < 1/\|\psi\|$. If

$$D(z_1, z_2) = (\psi(z_1) - \psi(z_2))/(z_1 - z_2),$$

then D is analytic in the bicylinder $U \times U$ and

$$|D(z_1, z_2)| \leqslant 2/(1-r)$$

in the bicylinder $U_r \times U$.

Then we have

$$z - t\psi(z) = z - \zeta(t) + t(\psi(\zeta(t)) - \psi(z))$$
$$= K(z, t)(z - \zeta(t)),$$

where

$$K(z, t) = 1 - tD(z, \psi(t)).$$

We see that K is analytic in $U \times U$ if $\|\psi\| < 1$ and

$$|K(z, t) - 1| \leqslant 2s/(1 - r) \quad \text{for } |z| \leqslant r, \qquad |t| \leqslant s.$$

This yields the representation

$$g(z, \lambda) = z^n + \lambda\phi = PG,$$

where

$$P(z, \lambda) = z^k \prod_1^{n-k} (z - \zeta(\omega^j \mu))$$

and

$$G(z, \lambda) = \prod_1^{n-k} K(z, \omega^j \mu).$$

We can obtain bounds on P and G for $|z| < 1$ and $|\lambda| < 1/\|\phi\|$, and on $1/G$ for $|z| + 2|\mu| \leqslant c < 1$.

Since equations of the type defining ζ are easy to handle, this representation may be useful when detailed information regarding ϕ is available.

IV. Generalization of the Weierstrass Preparation Theorem

The following considerations give, perhaps, a better insight into the meaning of this theorem. In the formulation of Theorem 3, we are given a small ϕ in R, and we seek a function $q \in R$ and a polynomial Q of degree $<n$ such that

$$z^n = (z^n + \phi)\, q - Q.$$

This is a special case of the representation of any $F \in R$ in the form

$$F = (z^n + \phi)q + \rho, \qquad q \in R, \qquad \deg \rho < n. \tag{5}$$

The general case follows from the special case on division of F by the polynomial $P = z^n + Q$.

On the other hand, when $\phi = 0$, the equation (5) has the unique solution

$$q = \Gamma_2 F, \qquad \rho = \Gamma_1 F = s_{n-1}(z, F).$$

The Weierstrass theorem says that (5) retains this property of solvability under the perturbation ϕ, and that q and ρ are analytic functions of ϕ.

The set $\mathscr{P}_{n-1}$ of polynomials of degree $<n$ is a closed module in the algebra R.

We are thus led to examine the general problem of a Banach algebra B and a closed module $M \subset B$. The element $x \in B$ is said to be M-regular, if every element $y \in B$ has a unique representation in the form

$$y = qx + r, \qquad q \in R, \qquad r \in M. \tag{6}$$

If M is the zero module $\{0\}$, then x is M-regular if and only if x has an inverse. A classical theorem (see, e.g. Gelfand, Raikov, and Shilov, [*4*], p. 20) states that the set of invertible elements is open and that x^{-1} is analytic on this set. We shall prove that the set of M-regular elements is open, and that q and r are analytic functions of x, for a general M. The special case $B = R$, $M = \mathscr{P}_{n-1}$, $x = x_n = z^n$, is essentially Theorem 3.

If X is M-regular, consider the Banach space $B \times M$, with the norm

$$\|(q,r)\| = \|q\| + \|r\|,$$

and the linear transformation

$$L(q,r) = qx + r.$$

This is continuous, and transforms $B \times M$ one-to-one onto B. Hence, by Banach [*1*], p. 41, L has a continuous inverse

$$L^{-1}(y) = (S_x(y), T_x(y)).$$

Let $\xi = x + \phi$, where $\|\phi\|$ is small. We wish to solve the equation

$$y = Q\xi + R.$$

This equation is equivalent to

$$y - Q\phi = Qx + R,$$

or

$$Q = S_x(y - Q\phi) = q - S_x(Q\phi),$$

and

$$R = T_x(y - Q\phi) = r - T_x(Q\phi).$$

The linear transformation $V(Q) = S_x(Q\phi)$ is a contraction if

$$\|V\| \leqslant s(x)\|\phi\| = k < 1.$$

Hence, if $\|\phi\| < 1/s(x)$, then there is a unique solution for Q:

$$Q = (I + V)^{-1}(q) = \sum_0^\infty (-V)^n(q) = V_1(q),$$

and, therefore, also a unique solution for R:

$$R = r - T_x(V_1(q)\phi).$$

We have the bounds

$$\|Q\| \leqslant \|q\|/(1-k) \leqslant s(x)\|y\|/(1-k),$$

and

$$\|R\| \leqslant t(x)\|y\|/(1-k).$$

These estimates imply that

$$\|Q - q\| \leqslant ks(x)\|y\|/(1-k),$$
$$\|R - r\| \leqslant kt(x)\|y\|/(1-k),$$
$$\|Q - q + S_x(q\phi)\| \leqslant k^2 s(x)\|y\|/(1-k),$$

and

$$\|R - r + T_x(q\phi)\| \leqslant k^2 t(x)\|y\|/(1-k),$$

The first two inequalities assert the continuity of S and T:

$$\|S_\xi - S_x\| \leqslant ks(x)/(1-k), \quad k = s(x)\|\xi - x\|, \tag{7}$$

and

$$\|T_\xi - T_x\| \leqslant kt(x)/(1-k),$$

while the last two assert that S and T are Fréchet differentiable at x. Let W and Ω be the linear transformations on B into $B_1 = B^B$, the space of bounded linear transformations on B to itself, defined by

$$W(\phi)(y) = -S_x(S_x(y)\phi),$$
$$\Omega(\phi)(y) = -T_x(S_x(y)\phi). \tag{8}$$

Then the Fréchet derivatives of S and T are W and Ω, respectively, and

$$\|S_\xi - S_x - W(\xi - x)\| \leqslant k_1 s(x)^3 \|\xi - x\|^2,$$
$$\|T_\xi - T_x - \Omega(\xi - x)\| \leqslant k_1 s(x)^2 t(x)\|\xi - x\|^2, \tag{9}$$

where

$$k_1 = 1/(1-k), \qquad k = s(x)\|\xi - x\| < 1.$$

We can summarize our results as follows:

THEOREM 4. *If M is a closed module in a Banach algebra B, then the set $R(M)$ of M-regular elements is open. If $x \in R(M)$, then the solutions $q = S_x(y)$ and*

$r = T_x(y)$ of (6) determine bounded linear transformations S_x and T_x on B into B and M, respectively. The sphere

$$\|\xi - x\| < 1/\|S_x\|$$

is contained in $R(M)$. If $k = s(x)\|\xi - x\| < 1$, $s(x) = \|S_x\|$, then S and T are continuous functions of x (inequalities (7)), and are, in fact, analytic on $R(M)$. Their Fréchet derivatives are W and Ω, given by formulas (8). The errors in approximating $S_\xi - S_x$ and $T_\xi - T_x$ by their first variations are estimated in inequalities (9).

APPENDIX

V. Some Other Criteria for Location of a Zero

Here we return to the question of detecting a zero of a function $f \in R$, $\|f\| \leqslant 1$, by means of the values of f at a few points. In Part I of this paper we showed that if $f(z_1)$ is not too small, and $f(z_2)$ is very small, where z_1 and z_2 are given in U, then f has a zero near z_2. In section II of the present part, we showed that if $f(z_1)$ is sufficiently small in comparison to $f'(z_1)$ (which is obtained from the values of f at two "infinitely near" points), then there is a zero of f near z_1. In these criteria we use the values of f at two points chosen in advance. In the present section, we give criteria for the existence of a zero near z_1 in terms of the values of $f(z_1)$ and $f(z_2)$, where the location of z_2 depends on the value of $f(z_1)$. Crudely speaking, if $|z_2 - z_1| \leqslant C|f(z_1)|$, and $|f(z_2)/f(z_1)|$ is too large or too small, then f has a zero near z_1. For example, if $|f(f(0))/f(0)|$ is greater than e^2 or less than $\exp(-2\eta)$, where

$$\log \eta + \eta + 1 = 0, \qquad 0 < \eta < 1,$$

then f has a zero near 0.

Suppose the $f \in R$, $\|f\| \leqslant 1$, and $|f(0)| = e^{-\alpha}$. If $f \neq 0$ in U_r, and $|z| = r_0$, $|f(z)| = e^{-\zeta}$, then, by Harnack's inequality, we have

$$\frac{r - r_0}{r + r_0}\alpha \leqslant \zeta \leqslant \frac{r + r_0}{r - r_0}\alpha,$$

or

$$r \leqslant r_0/t_0,$$

where

$$t_0 = |\zeta - \alpha|/(\zeta + \alpha). \tag{10}$$

Hence, if $r_0 < t_0$, we infer that f has a zero in U, and obtain the non-trivial bound r_0/t_0 for the smallest zero.

The condition $r_0 < t_0$ is equivalent to

$$\min(\zeta/\alpha, \alpha/\zeta) < (1 - r_0)/(1 + r_0).$$

This form of the condition is useful if we are given in advance $r_0 = |z|$, where z is the second point where f is computed. We are interested here in the situation where r_0 depends on α, i.e., we compute $f(0)$ and, depending on its value, we choose the point z at which we compute f. In this case, it is more convenient to put the criterion in the form

$$\alpha - \zeta > 2r_0 \alpha/(1 + r_0) \qquad \text{or} \qquad \zeta - \alpha > 2r_0 \alpha/(1 - r_0).$$

For example, let $r_0 = c|f(0)|^k = c\exp(-k\alpha)$, where $c > 0$, $k > 0$. Then we have

$$2r_0 \alpha/(1 + r_0) = 2c/kh(y),$$

where

$$h(y) = (e^y + c)/y, \qquad y = k\alpha.$$

The minimum of $h(y)$ for $y > 0$ is attained at $1 + \eta$, where

$$\eta \exp(\eta + 1) = c, \tag{11}$$

the minimum being c/η. Therefore, we shall have $r_0 < t_0$, if $\zeta - \alpha > 2\eta/k$.

Similarly, we find that if $0 < c \leqslant 1$, then the maximum of $2r_0 \alpha/(1 - r_0)$ is $2\gamma/k$, where γ is the solution of the equation

$$\gamma \exp(1 - \gamma) = c. \tag{12}$$

We have thus proved

THEOREM 5. *If* $f \in U$, $\|f\| \leqslant 1$, $|f(0)| = e^{-\alpha}$, $|z_1| \leqslant c|f(0)|^k = c\exp(-k\alpha)$, $c > 0, k > 0$, *and* $|f(z_1)| = e^{-\zeta}$, *and if* $f(z) \neq 0$ *in the circle* $|z| < |z_1|/t_0$, *where* t_0 *is given by* (10), *then*

$$\alpha - \zeta \leqslant 2\eta(c)/k,$$

where $\eta(c)$ *is the solution of* (11), *and if* $0 < c \leqslant 1$, *then*

$$\zeta - \alpha \leqslant 2\gamma(c)/k,$$

where $\gamma(c)$ *is the solution of* (12).

If we take $c = 1$, $k = 1$, then we obtain

COROLLARY 5a. *If* $f \in R$, $\|f\| \leqslant 1$, $|z_1| \leqslant |f(0)|$, *and if* $f(z) \neq 0$ *in* $|z| < |z_1|/t_0$, *then*

$$-2\eta(1) \leqslant \log|f(z_1)/f(0)| \leqslant 2,$$

VI. Newton's Method

Corollary 1a, applied to $g(z) = a_0 + f(z) = a_0 + zh(z)$, $g'(0) = a_1 = h(0)$, states that if $|a_0| < \|h\|\,\alpha(|a_1|/\|h\|)$, then g has a unique small zero $z(g)$, and that

$$\left| z(g) + \frac{g(0)}{g'(0)} \right| \leqslant 12|a_0|^2\,\|h\|/|a_1|.$$

We recognize the approximation formula

$$z(g) \sim -g(0)/g'(0) = H(0)$$

as the first step in Newton's method:

$$z(g) \sim H(z) = z - g(z)/g'(z),$$

That is, our corollary gives us a criterion for the existence of a unique small zero $z(g)$, and an estimate for the error $|z(g) - H(0)|$ in the first step of Newton's method.

This raises the question of the behavior of the iterates of H, and of the domain of convergence of Newton's method. Of course, there is a vast literature on this subject (see Ostrowski [7]).

We wish to discuss here the domain of attraction of H around a fixed point, which is, of course, a zero of g.

More precisely, given $f \in R$, $\|f\| \leqslant 1$, $f(0) = 0$, $|f'(0)| = A > 0$, we wish to determine an $r \leqslant 1$ such that for $|z| \leqslant r$,

$$H(z) = z - f(z)/f'(z)$$

satisfies $|H(z)| \leqslant |z|$, and, more generally, for $0 < k \leqslant 1$, to find $r(k)$ such that $|H(z)| \leqslant k|z|$ in the circle $|z| \leqslant r(k)$.

We may, without loss of generality, assume that $f'(0) = A$. Then f can be represented in the form

$$f(z) = z\phi(v) = zh(z),$$

where

$$\phi(z) = (A - z)/(1 - Az),$$

and $v \in R$, $\|v\| \leqslant 1$, $v(0) = 0$. We find that

$$H(z)/z = u/(1 + u),$$

where $u(z) = zh'(z)/h(z)$.

We wish to determine the domain of variation of $u(z)$ for a fixed z, $|z| = r < A$, as v ranges over the set E:

$$v \in R, \qquad \|v\| \leqslant 1, \qquad v(0) = 0.$$

Now, we have

$$u = z\phi'(v)\,v'/\phi(v) = -(1 - A^2)\,zv'/\psi(v),$$

where

$$\psi(v) = (A - v)(1 - Av).$$

But it is known (see Heins [5], p. 84) that

$$|v'(z)| \leqslant (1 - |v(z)|^2)/(1 - r^2),$$

and that for given z and $v(z)$, $v'(z)$ can take on any value in this circle. Hence, given z and $v(z)$, $u(z)$ can take on any value in the circle

$$|u(z)| \leqslant (1 - A^2)\, r (1 - |v(z)|^2)/\psi(|v(z)|)(1 - r^2).$$

We have

$$\psi(x)^2 \frac{d}{dx}((1 - x^2)/\psi(x)) = (1 + A^2)\left(\left(x - \frac{2A}{1 + A^2}\right)^2 + \left(\frac{1 - A^2}{1 + A^2}\right)^2\right) > 0,$$

so that $(1 - x^2)/\psi(x)$ is increasing. But, by Schwarz' lemma, $|v(z)| \leqslant r$, and $v(z)$ can take on any value in U_r. Consequently, we have

$$|u(z)| \leqslant (1 - A^2)\, r/\psi(r) = -r\phi'(r)/\phi(r) = U(r)$$

for $|z| \leqslant r < A$.

This inequality is the best possible, and the equality is attained for $v(z) = cz$, $|c| = 1$,

$$f(z) = z\phi(cz) = \mathfrak{F}(cz)/c,$$

where $\mathfrak{F}(z) = z\phi(z)$. If

$$\mathfrak{H}(z) = z - \mathfrak{F}(z)/\mathfrak{F}'(z)$$

is the "Newton function" corresponding to $\mathfrak{F}$, then we can express this result in the form:

$$|H(z)/(z - H(z))| \leqslant |\mathfrak{H}(r)/(r - \mathfrak{H}(r))| \qquad \text{for } |z| \leqslant r < A.$$

Hence, if $U(r) < 1$, then

$$|H(z)/z| \leqslant |\mathfrak{H}(r)|/r,$$

and this is true for $r < r_0 = A/(1 + (1 - A^2)^{1/2})$. Since $\mathfrak{H}$ has a pole at r_0, therefore for the class of functions considered, there is no uniform bound for H in any circle U_r, $r \geqslant r_0$.

We find that for $0 < k \leqslant 1$, $r(k)$ is the root of the equation

$$U(r) = k/(k + 1) \qquad \text{or} \qquad \mathfrak{H}(r) = -kr, \tag{13}$$

and we obtain

$$r(k) = A(\sigma - 1)/(\sigma + 2k + 1), \tag{14}$$

where

$$\sigma = \{((2k + 1)^2 - A^2)/(1 - A^2)\}^{1/2}.$$

We thus have

THEOREM 6. *Let $E(A)$ be the set of $f \in E$, such that $f(0) = 0$ and $|f'(0)| = A$. For $f \in E(A)$, let*

$$H(z) = z - f(z)/f'(z) = H(z, f).$$

Then for $|z| \leqslant r < r_0 = A/(1 + (1 - A^2)^{1/2})$, we have

$$\max_{f \in E(A)} |H(z)| = |\mathfrak{H}(r)|,$$

where $\mathfrak{H}(z) = \mathfrak{H}(z, \mathfrak{F})$, $\mathfrak{F}(z) = z(A - z)/(1 - Az)$. For $|z| \geqslant r_0$, $H(z, f)$ is unbounded for $f \in E(A)$. For $0 < k \leqslant 1$, $|H(z, f)| \leqslant k|z|$ for $|z| \leqslant r(k)$, where $r(k)$ is given by (13), (14), and this is the best possible. Hence, the iterates of H converge to zero for $|z| < r(1)$.

For $r_0 > r > r_1 = A/(1 + (1 + A)(1 - A)^{1/2})$, we have $|\mathfrak{H}(r)| > 1$; hence there are f's in $E(A)$ such that for given z, $|z| = r > r_1$, $|H(z)| > 1$, so that $H_2(z) = H(H(z))$ may be undefined. For $r(1) < |z| < r_1$, our theorem does not tell us anything about the convergence of $H_n(z)$.

For $f = \mathfrak{F}$, $H = \mathfrak{H}$, if we set

$$\mu(z) = (1 - A^2)\, z/(A - z),$$

then we have

$$\mu(\mathfrak{H}(z)) = -\mu(z)^2$$

and therefore

$$\mu(\mathfrak{H}_n(z)) = -\mu(z)^{2^n}.$$

Hence $\mathfrak{H}_n(z) \to 0$ in the region $|\mu(z)| < 1$, and $\mathfrak{H}_n(A) \to A$ in the circle $|\mu(z)| > 1$. In general, for $|\mu(z)| = 1$, $\mathfrak{H}_n(z)$ diverges.

The region $|\mu(z)| < 1$ is the exterior of the circle with diametral points at $A/(2 - A^2)$ and $1/A$. The circle $|z| < A/(2 - A^2)$ is the largest circle with center at the origin contained in this region. Therefore $\mathfrak{H}_n(z)$ converges in this circle which is larger than $|z| < r(1)$.

We note that if $|z| < r$, $U(r) = \lambda$, then $H(z)/z$ lies in the image of the circle U_λ under the mapping

$$w = u/(1 + u).$$

This is the interior of the circle with diametral points at $\lambda/(1 + \lambda)$ and $-\lambda/(1 - \lambda)$ for $\lambda < 1$, the exterior of this circle for $\lambda > 1$, and the half-plane $\Re(w) < 1/2$ for $\lambda = 1$.

We can obtain, more generally, an estimate for

$$\frac{H(z)}{\lambda z - H(z)} = \frac{u(z)}{\lambda + (\lambda - 1)\, u(z)}$$

if $\lambda > 1$. For then we have

$$|H(z)/(\lambda z - H(z))| \leqslant U(r)/(\lambda - (\lambda - 1)\, U(r))$$

if $U(r) < \lambda/(\lambda - 1)$. But the right-hand side is easily expressible in terms of $\mathfrak{H}$, and we infer that

$$|H(z)/(\lambda z - H(z))| \leqslant |\mathfrak{H}(r)/(\lambda r - \mathfrak{H}(r))|,$$

under the condition $U(r) < \lambda/(\lambda - 1)$. If we set $\lambda = A/r$, and observe that $U(r) < A/(A - r)$ for $r < A$, we conclude that

$$\left|\frac{rH(z)}{Az - rH(z)}\right| \leqslant \left|\frac{\mathfrak{H}(r)}{A - \mathfrak{H}(r)}\right| = \frac{\mu(r)^2}{1 - A^2}.$$

We have in particular,

$$|\mu(H(r))| \leqslant \mu(r)^2 \qquad \text{for} \quad 0 \leqslant r < A.$$

If f is such that $H(r)$ is real and non-negative for $0 \leqslant r < A$, then we see that

$$\mu(H_n(r)) \leqslant \mu(r)^{2^n}$$

and therefore $H_n(r) \to 0$ if $\mu(r) < 1$, i.e., $r < A/(2 - A^2)$.

Thus, under these additional assumptions, Newton's method converges on the same interval $[0, A/(2 - A^2))$ as it does for the special function $\mathfrak{F}$, and this result cannot be improved. It would be desirable to clear up the question of the behavior of $H_n(z)$ in the annulus $r(1) < |z| < r_1$, for general $f \in E(A)$.

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